Mathematics Methods Unit 2 Test 5 2023

42

Name: 17/10/23 **Total marks: 55**

23

Calculator free **Time Allowed:** 25 minutes **Total for this section:** 24 marks

Instructions: Answer in the spaces provided. Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. Correct answers given without supporting reasoning may not be awarded full marks.

Question 1. **(4 marks)**

(a) Determine

(i) $\frac{d}{dx}((x-7)(5x-2))$. (1 mark)

$= \frac{d}{dx}(5x^2 - 2x - 35x + 14)$

$= \frac{d}{dx}(5x^2 - 37x + 14)$

$= 10x - 37$

(ii) $\frac{d}{dx}\left(\frac{x^5 - x^2}{5x}\right)$ $= \frac{d}{dx}\left(\frac{x^5}{5x} - \frac{x^2}{5x}\right)$ (1 mark)

$= \frac{d}{dx}\left(\frac{x^4}{5} - \frac{x}{5}\right)$

$= \frac{4x^3}{5} - \frac{1}{5}$ $= \frac{4x^3 - 1}{5}$

(b) Find $\frac{dy}{dx}$ for $y = \frac{1}{x} - \sqrt{3x^5}$ (2 marks)

$y = x^{-1} - (3x^5)^{\frac{1}{2}}$

$y = x^{-1} - \sqrt{3}\, x^{\frac{5}{2}}$

$\frac{dy}{dx} = -x^{-2} - \frac{5\sqrt{3}}{2} x^{\frac{3}{2}}$

$= -\frac{1}{x^2} - \frac{5\sqrt{3}\, x^{\frac{3}{2}}}{2}$

4

(3 marks)

Question 2.

Use the first principles to obtain $\frac{dy}{dx}$ if $y = 3x - 2x^2$

$$y = 3x - 2x^2 \Rightarrow f(x) = 3x - 2x^2,\ f(x+h) = [3(x+h) - 2(x+h)^2]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[3(x+h) - 2(x+h)^2] - (3x - 2x^2)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - 2(x^2 + 2xh + h^2) - 3x + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x} + 3h \cancel{- 2x^2} + 4xh + 2h^2 \cancel{- 3x} \cancel{+ 2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{3h + 4xh + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3 + 4x + 2h)}{\cancel{h}}$$

$$= 3 + 4x \text{ as } h \to 0$$

2

uestion 3.

(5 marks)

(a) Find the anti-derivative of $(2r-1)^2$ (1 mark)

$(2r-1)(2r-1) = 4r^2 - 4r + 1$ $\quad \frac{ax^{n+1}}{n+1}$

$f'(x) = \frac{4r^3}{3} - 2r^2 + r + c$

(b) Find the following indefinite integrals.

(i) $\int \frac{x^3}{6} + 9\,dx$ $= \int \frac{1}{6}x^3 + 9\,dx$ (2 marks)

$= \frac{\frac{1}{6}x^4}{4} + 9x + c$

$= \frac{x^4}{24} + 9x + c$

(ii) $\int x(2-3x)dx$ $= \int 2x - 3x^2\,dx$ (2 marks)

$= x^2 - x^3 + c$

Question 4. (3 marks)

Determine the equation of the tangent to the curve $y = x^2 - 5x$ at $x = 1$.

$\frac{dy}{dx} = 2x - 5$

at $x = 1$, $y = 1 - 5$
$= -4$

$\left.\frac{dy}{dx}\right|_{x=1} = 2 - 5$
$= -3$

$y = mx + c$

$\therefore -4 = -3(1) + c$

$-4 = -3 + c$

$-1 = c$

$\therefore y = -3x - 1$

8

Question 5. (3 ma

Given that $\frac{dv}{dt} = kt - 5$, where k is a constant, find v if $v(0) = 2$ and $v(2) = -10$.

when $t = 0,\ v = 2$

$t = 2,\ v = -10$

$$\frac{dv}{dt} = kt - 5$$

$$\int (kt - 5)\, dt = \frac{kt^2}{2} - 5t + c$$

$$2 \cancel{-10} = \frac{k(0)^2}{2} - 5(0) + c$$

$$c = 2$$

$$\therefore -10 = \frac{k(2)^2}{2} - 10 + 2$$

$$-10 = \frac{4k}{2} - 10 + 2$$

$$-10 = 2k - 10 + 2$$

$$0 = 2k + 2$$

$$0 = k + 1$$

$$k = -1$$

$$\frac{dv}{dt} = -t + 5$$

$$v = \frac{-t^2}{2} - 5t + 2$$

3

uestion 6

Let $f(x) = x^3 - 3x^2 + 4$. **(6 marks)**

(a) Determine the coordinates of the stationary points of $f(x)$. (2 marks)

$f'(x) = 3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or 2

at $x = 0$, $y = 4$

at $x = 2$, $y = 8 - 3(4) + 4$

$= 8 - 12 + 4$

$= 0$

$\therefore$ Stationary pts at $(0,4)$, $(2,0)$

(b) Using the sign test, determine the nature of the stationary points of the curve $y = f(x)$. (2 marks)

x	$x<0$	$x=0$	$x>0$
$f'(x)$	$+$	0	$-$
	/	—	\

$(-1)^2 - 2(-1)$
$= 1 + 2$
$= 3$

$1^2 - 2(1)$
$= -1$

$\therefore (0,4)$ is a max TP

x	$x<2$	$x=2$	$x>2$
$f'(x)$	$-$	0	$+$
	\	—	/

$3^2 - 2(3)$

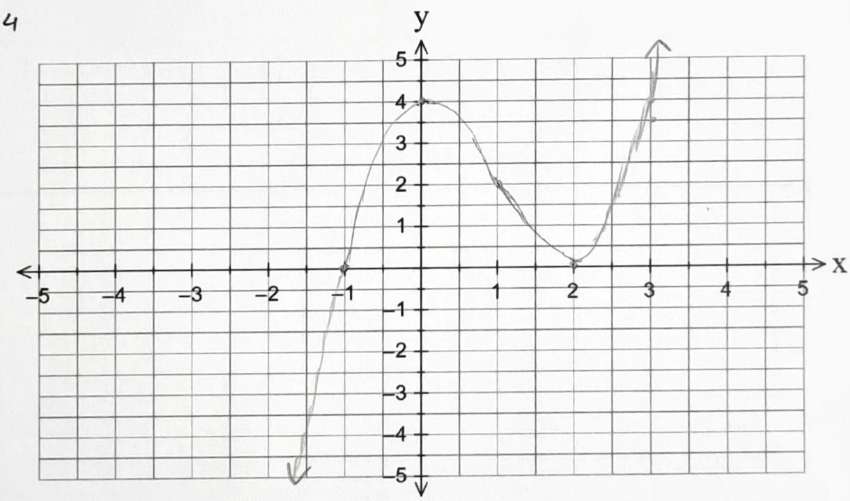
$\therefore (2,0)$ is a min TP

(c) Given that $f(-1) = 0$, sketch the graph of $y = f(x)$, indicating all stationary points. (2 marks)

$3^3 - 3(3)^2 + 4$
$27 - 27 + 4$

$(-2)^3 - 3(-2)^2 + 4$
$= -8 - 12 + 4$
$=$

$1^3 - 3(1)^2 + 4$
$= 1 - 3 + 4$
$= 2$



6

Mathematics Methods Unit 2 Test 5 2023

Name:

Calculator Assumed.

Time Allowed: 30 minutes **Total for this section:** 19 31 marks.

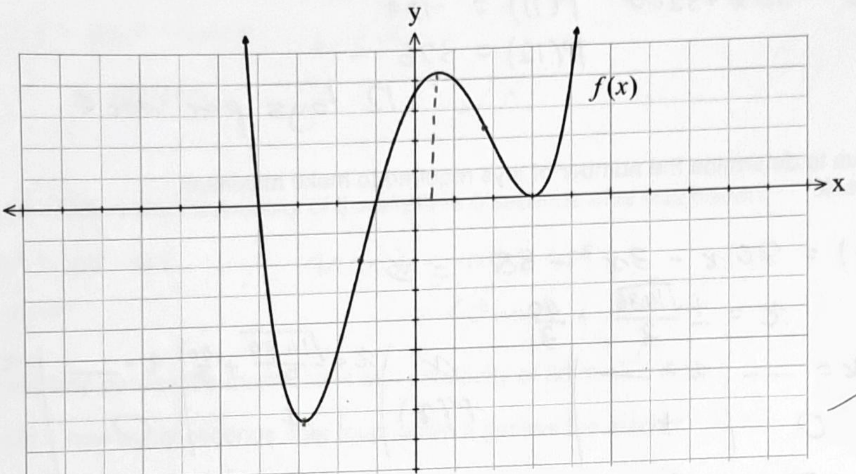
Instructions:

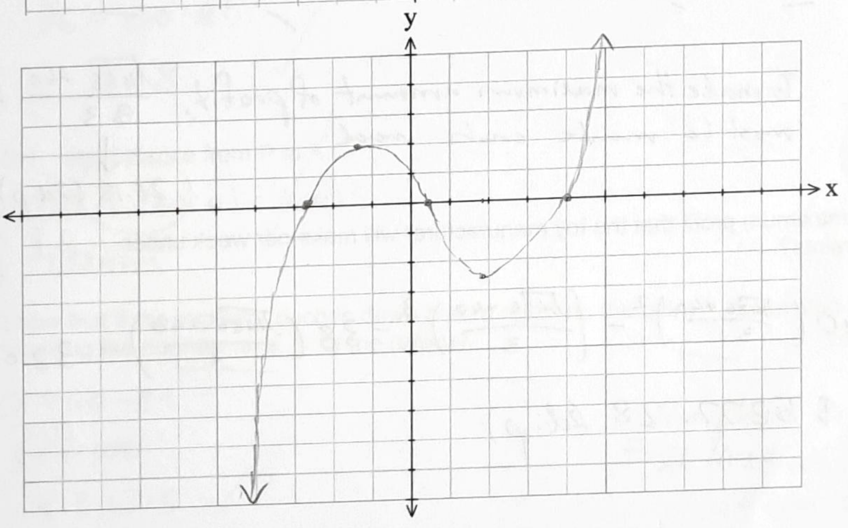
Answer in the spaces provided. Show all working clearly, in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. Correct answers given without supporting reasoning may not be awarded full marks.

Question 7. **(2 marks)**

The graph of $f(x)$ is given below.

Sketch a possible graph of $f'(x)$, on the set of axes given below the graph of $f(x)$.





2

(6 m

Question 8

A toy manufacturer produces x toys per week.

The cost of producing each toy is \$42, and the initial cost of setting up the machinery is \$3200.

The revenue raised from the sale of those x toys is given as $R(x) = 40x^2 - x^3 + 4x$.

To determine the profit, the cost is subtracted from the revenue.

(a) Show that the profit $p(x)$ is given as $P(x) = 40x^2 - x^3 - 38x - 3200$. (1 mark)

+

$P(x) = 40x^2 - x^3 + 4x - \cancel{42(x)}\ 42x - 3200$

$= 40x^2 - x^3 - 38x - 3200$

(b) How many toys per week need to be made so that a profit is produced? (1 mark)

~~solve $240x^2 > 2x^3 + 38x + 3200$~~

$P(11) = -109$

$P(12) = 376$

$\therefore$ 12 toys per week

(c) Use Calculus to determine the number of toys required to make maximum profit per week. 2 ~~(3 marks)~~

$\frac{dP}{dx}$ $P'(x) = 80x - 3x^2 - 38 = 0$

$x = \pm\frac{\sqrt{1486}}{3} + \frac{40}{3}$

x	$x < \frac{-\sqrt{1486}}{3} + \frac{40}{3}$	$x =$ —	$x >$ —
$P'(x)$	$-$	0	+
	\	—	/

x	$x < \frac{\sqrt{1486}}{3} + \frac{40}{3}$	$x =$ —	$x >$ —
$P'(x)$	+	0	$-$
	/	—	\

$\therefore$ To make the maximum amount of profit, $\frac{\sqrt{1486} + 40}{3}$ toys must be made each week

26.18 (2d.p)

27 or 26 toys

(d) What is the maximum profit that the toy manufacturer will make per week under these constraints? (1 mark)

$P(x) = 40\left(\frac{\sqrt{1486}+40}{3}\right)^2 - \left(\frac{\sqrt{1486}+40}{3}\right)^3 - 38\left(\frac{\sqrt{1486}+40}{3}\right) - 3200$

$= \$5277.28$ (2d.p)

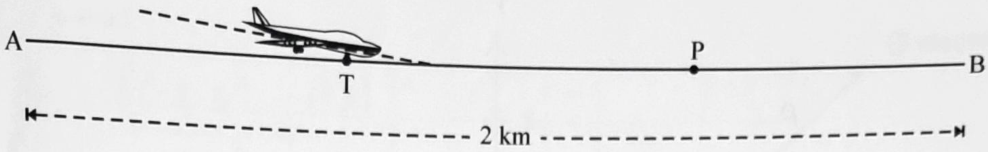
f/t

5

estion 9

(8 marks)

he main runway at Concordville airport is 2 km long. An airplane, landing at Concordville, touches down at point T, and immediately starts to slow down. The point A is at the southern end of the runway. A marker is located at point P on the runway.



Not to scale

As the airplane slows down, its distance, *s*, from A, is given by $s = c + 100t - 4t^2$,

where *t* is the time in seconds after touchdown, and *c* metres is the distance of T from A.

(a) The airplane touches down 800 m from A, (*ie* c = 800). (2, 2, 1, 1 marks)

(i) Find the distance travelled by the airplane in the first 5 seconds after touchdown.

$S = 800 + 100(5) - 4(5)^2$

$= 1200\text{m}$

(ii) Determine the velocity of the airplane 5 seconds after touchdown.

$v = 100 - \cancel{8}t$ $\therefore v(5) = 100 - 40$

$= 60 \text{m/s}$

The airplane passes the marker at P with a velocity of 36 m s^{-1}. Find

(iii) how many seconds after touchdown it passes the marker.

$36 = 100 - 8t$

$t = 8$

(iv) the distance from P to A.

$d = 800 + 100(8) - 4(8)^2$

$= 1344\text{m}$

(b) Show that if the airplane touches down before reaching the point P, it can stop before reaching the northern end, B, of the runway. (2 marks)

$0 = 100 - 8t$

$8t = 100$

$t = 12.5$

$d = 800 + 100(12.5) - 4(12.5)^2$

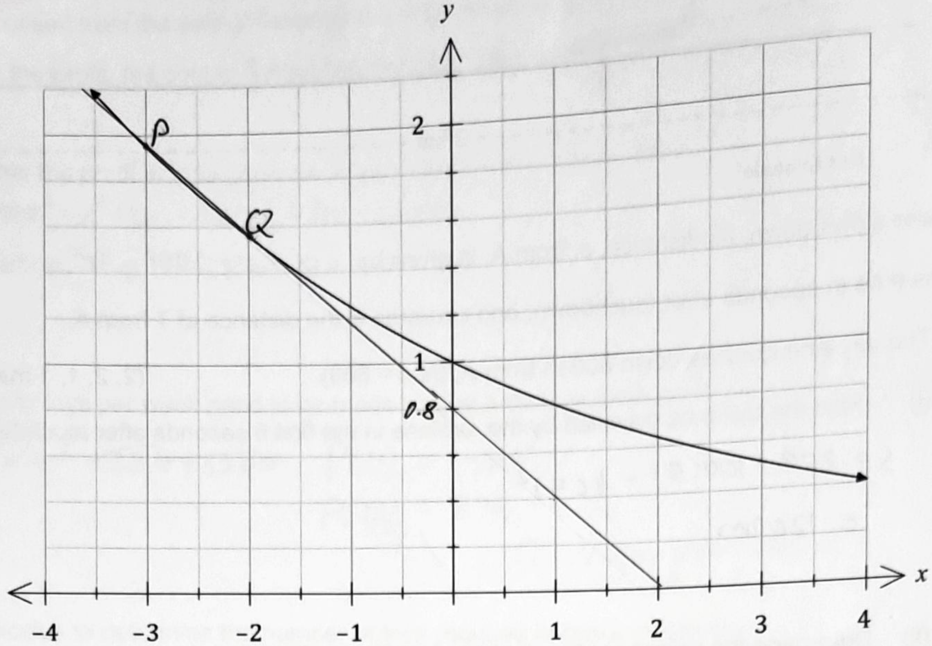
$= 1425$

$\therefore$ The plane is stationary at 1425m from A, $\therefore$ it can stop before reaching point B.

6

Question 10 **(8 marks)**

A function is defined by $f(x) = 0.8^x$ and the graph of $y = f(x)$ is shown below.



(a) Draw the chord between the points P and Q on the curve $y = f(x)$ that have x-coordinates -3 and 2 respectively and determine the slope of this chord. (3 marks)

$y = mx + c$

$0 = 2m + 0.8$

$-0.8 = 2m$

$m = -0.4$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0.8^2 - 0.8^{-3}}{2-(-3)}$

$= -0.263$

point R with x-coordinate $-3+h$ lies on the curve between P and Q, where $h > 0$.

(b) Use the difference quotient $\frac{\delta y}{\delta x} = \frac{f(x+h)-f(x)}{h}$ to calculate the slope of chord PR when

(i) $h = 0.5$.

(2 marks)

$$\frac{dy}{dx} = \frac{f(-2.5) - f(3)}{0.5} = \frac{0.8^{-3+0.5} - 0.8^{-3}}{0.5}$$

$$= \frac{1.8 - 2}{0.5}$$

$$= \frac{-0.2}{0.5} = -0.4$$

(ii) $h = 0.1$.

(1 mark)

$$\frac{dy}{dx} = \frac{0.8^{-3+0.1} - 0.8^{-3}}{0.1}$$

(c) Show use of the difference quotient to determine the slope of tangent to the curve at P that is correct to 3 decimal places.

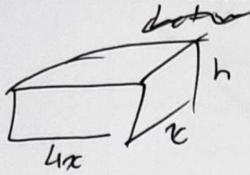
(2 marks)

(7 marks)

Question 11

The length of a rectangular prism is four times its width, and the sum of its height, width and length is 210 cm. Let the width of the cuboid be x cm.

(a) Show that the volume of the rectangular prism is $840x^2 - 20x^3$ cm^3. (2 marks)



$4x$ $5x + h = 210$

$x = -0.2h + 42$

$h = -5x + 210$

$\therefore V = 4x(x)(-5x+210)$

$= 840x^2 - 20x^3$

(b) Use a method involving differentiation to determine the height of the rectangular prism that maximises its volume. (2 marks)

$\frac{dV}{dx} = 1680x - 60x^2 = 0$

$x = 28, 0 \Rightarrow 0$ is discarded

$\therefore h = -5(28) + 210$

$= 70$ cm

(c) Determine the maximum possible total surface area of the rectangular prism. (3 marks)

$x = -0.2(70) + 42$

$= 28$

$SA = 28(28)(4)(2) + 28(70)(2) + 4(28)(70)(2)$

$= 25872$ cm^2

End of Test 5

5